Analysis 1 HW 4

2.6

Definitions
$E \subset X$.
E' is the set of all limit points of E.
$\bar{E} = E \cup E'$.

Prove that E' is closed.

Proof. To show E' is closed, we will show that E' contains all of its limit points. Let p be a limit point E'. We will show that p is a limit point of E, which in turn implies $p \in E'$. Let $\epsilon > 0$ be arbitrary. $B_{\frac{\epsilon}{2}}(p) \cap E'$ contains at least one other point q other than p. Also, since q is a limit point of E, $B_{\frac{\epsilon}{2}}(q) \cap E$ contains at least one point r other than q.

$$\begin{aligned} d(p,r) &\leq d(p,q) + d(q,r) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So, for every $\epsilon > 0$, we have $r \in B_\epsilon(p) \cap E$, $r \neq p$. Therefore, p is a limit point of E. □

Prove that E and $\bar{E}$ have the same limit points.

Proof. It is clear that if p is a limit point of E, it must also be a limit point of $\bar{E}$. Going the other way, let p be a limit point of $\bar{E}$. Let $\epsilon > 0$ be arbitrary. Then, $B_{\frac{\epsilon}{2}}(p) \cap \bar{E}$ contains at least one q other than p. If $q \in E$, we are done. If $q \notin E$, it must be the case that $q \in E'$, i.e, q is a limit point of E. Thus, $B_{\frac{\epsilon}{2}}(q) \cap E$ must contain a point r other than q. Now, $d(p,r) \leq d(p,q) + d(q,r) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So, E and $\bar{E}$ have the same limit points. □

Do E and E' always have the same limit points?

No. Consider $E = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$. $E' = \{0\}$. The set of limit points of E' is the null set.

2.29

Prove that every open set in $\mathbb{R}^1$ is a union of an at most countable collection of disjoint segments.

Proof. Let U be an open set in $\mathbb{R}$. Let $x, y, z \in U$. We define $x \sim y$ if ($x \leq y$ and $[x,y] \subset U$) or ($y < x$ and $[y,x] \subset U$). This relation is

1. reflexive, since $x \sim x$ ($[x, x] \subset U$),
2. symmetric, since $x \sim y \implies y \sim x$, and
3. transitive, since $x \sim y$ and $y \sim z \implies x \sim z$.

Thus, this relation induces a partition P on U. We know that all elements of a partition are disjoint. To show that every $Q \in P$ is a segment, consider $p \in Q$. Since U is open, $r > 0$ exists such that $B_r(p) \subset U$. For every $b \in B_r(p)$, if $b \leq r$ we have $[b, r] \subset B_r(p) \subset U$, and if $b > r$, we have $[r, b] \subset B_r(p) \subset U$.So, $b \sim p$, i.e, b and p must belong to the same equivalence class, Q. It follows that $B_r(p) \subset Q$. Therefore, each $Q \in P$ much be a segment in $\mathbb{R}$.

Now, for every $Q_i \in P$, define $q_i \equiv (\inf Q_i + \sup Q_i)/2$. Since Q_i are disjoint, each Q_i has a unique q_i associated with it. Thus, we have $P \sim \mathbb{Q}' \subset \mathbb{Q}$. Since $\mathbb{Q}$ is countable, P is at most countable. □

4.2

Definitions
$f : X \to Y$ is continuous, where X and Y are metric spaces.

Prove that $f(\overline{E}) \subset \overline{f(E)}$ for every set $E \subset X$.

Proof. Let $p \in f(\overline{E})$. Then, there exists a q in $\overline{E}$ such that $f(q) = p$. If $q \in E$, we are done. If $q \notin E$, q must be a limit point of E. So, there exists a sequence (q_n) in E which converges to q. Since f is continuous, $(f(q_n))$ must converge to $f(q)$. Thus, $f(q)$ is a limit point of $f(E)$, and so $f(q) = p \in \overline{f(E)}$. □

Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Consider $f : \mathbb{N} \to \mathbb{R}$, $f(n) = \frac{1}{n}$. Since $\mathbb{N}$ consists of only isolated points, f is continuous. Also, $\overline{\mathbb{N}} = \mathbb{N}$, since $\mathbb{N}$ is a closed set. Thus, $f(\mathbb{N}) = f(\overline{\mathbb{N}}) = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. However, 0 is a limit point of $f(\mathbb{N})$. Thus, $0 \in \overline{f(\mathbb{N})}$, but $0 \notin f(\overline{\mathbb{N}})$.

4.4

Definitions
$f, g : X \to Y$ are continuous, where X and Y are metric spaces.
$E \subset X$ is dense in X.

Prove that $f(E)$ is dense in $f(X)$.

Proof. Let $p \in f(X)$. There exists $q \in X$ such that $f(q) = p$. Since E is dense in X, either $q \in E$ or q is a limit point of E. If $q \in E$, $f(q) \in f(E)$. If q is a limit point of E, there exists a sequence (q_n) in E which converges to q. Since f is continuous, $(f(q_n))$ must converge to $f(q)$. Thus, $f(q)$ is a limit point of $f(E)$. Therefore, either $p \in f(E)$, or p is a limit point of $f(E)$, i.e, $f(E)$ is dense in $f(X)$. □

If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$.

Proof. Again, E being dense in X tells us that either $p \in E$ or p is a limit point of E. If $p \in E$, the result is true by hypothesis. In the latter case, there exists a sequence (p_n) in E that converges to p. Since f and g are continuous, $(f(p_n)) \to f(p)$ and $(g(p_n)) \to g(p)$. Also, $f(p_n) = g(p_n)$ for all $n \in \mathbb{N}$, by hypothesis. Since the limit of a sequence is unique, it follows that $f(p) = g(p)$. □

4.7

Definitions

$f, g : \mathbb{R}^2 \to \mathbb{R}$.

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{xy^2}{x^2+y^4} & (x,y) \neq (0,0) \end{cases}$$

$$g(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{xy^2}{x^2+y^6} & (x,y) \neq (0,0) \end{cases}$$

Prove that f is bounded on $\mathbb{R}^2$.

Proof. We will show that

$$\left| \frac{xy^2}{x^2+y^4} \right| \leq \frac{1}{2}$$

for all $(x,y) \in \mathbb{R}^2$. On applying the AM-GM inequality on x^2 and y^4, we get

$$\frac{x^2+y^4}{2} \geq \sqrt{x^2y^4} = |x|y^2$$
$$\implies \frac{|x|y^2}{x^2+y^4} = \left| \frac{xy^2}{x^2+y^4} \right| \leq \frac{1}{2}.$$

□

Prove that g is unbounded on every neighborhood of $(0,0)$.

Proof. Let $\epsilon > 0$. We will show that there exists $(x,y) \in B_\epsilon((0,0), \mathbb{R}^2)$ such that

$$|g(x,y)| = \left| \frac{xy^2}{x^2+y^6} \right| > M \tag{1}$$

for all $M > 0$. Assume $x, y > 0$.

$$\begin{aligned} \frac{xy^2}{x^2+y^6} &> M \\ \implies \frac{x^2+y^6}{xy^2} = \frac{x}{y^2} + \frac{y^4}{x} &< \frac{1}{M} \end{aligned}$$

Define $\alpha \equiv x/y^2$.

$$\alpha + \frac{y^2}{\alpha} < \frac{1}{M}.$$

Let $\alpha = \frac{1}{2M}$. Let us attempt to solve for y.

$$\begin{aligned} \frac{1}{2M} + 2My^2 &< \frac{1}{M} \\ y &< \frac{1}{2M} \end{aligned}$$

Setting $y = \frac{1}{4M}$ will suffice. Now, we can solve for x:

$$x = \frac{y^2}{2M} = \frac{1}{32M^3}$$

Thus, $\left(\frac{1}{32M^3}, \frac{1}{4M}\right)$ will satisfy (1). This can be easily verified:

$$g\left(\frac{1}{32M^3}, \frac{1}{4M}\right) = \frac{\left(\frac{1}{4M}\right)^2\left(\frac{1}{32M^3}\right)}{\left(\frac{1}{4M}\right)^6 + \left(\frac{1}{32M^3}\right)^2} = \frac{8M}{5} > M.$$

Now, if $|x| < \epsilon/\sqrt{2}$ and $|y| < \epsilon/\sqrt{2}$, it must be that $(x, y) \in B_\epsilon(0, 0)$. There exist $k_1 \geq 1$ and $k_2 \geq 1$ such that $\frac{1}{32(k_1 M)^3} < \frac{\epsilon}{\sqrt{2}}$ and $\frac{1}{4k_2 M} < \frac{\epsilon}{\sqrt{2}}$. Let k be the greater of the two. Since being greater than kM ensures being greater than M, we have found the (x, y) we were in the search of:

$$(x, y) = \left(\frac{1}{32(kM)^3}, \frac{1}{4kM}\right).$$

□

Prove that f is not continuous at $(0, 0)$.

Proof. Since $(0, 0)$ is a limit point of $\mathbb{R}^2$, we must have $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 0$ for f to be continuous at $(0, 0)$. If we plug $x = y^2$ into the definition of f, we get $\frac{1}{2}$. Thus, for $\epsilon < \frac{1}{2}$, it is not possible to find a $\delta > 0$ such that $|(x, y)| < \delta \implies |f(x, y)| < \epsilon$, since we will always be able to find $(a, b) \in B_\delta(0, 0)$ such that $a = b^2$. □