Analysis 1 HW 3

1

Definitions
(x_n) is a sequence of real numbers satisfying $3x_{n+1} = x_n^3 - 2$ for all $n \in \mathbb{N}$.

Solution
If $(x_n) \to l$ for some l, then

$$\lim_{n\to\infty}(3x_{n+1}) = \lim_{n\to\infty}(x_n^3 - 2)$$
$$\implies 3l = l^3 - 2$$

courtesy the algebraic limit theorem. This yields $l \in \{2, -1\}$.

Now, consider the successive differences:

$$x_{n+1} - x_n = \frac{x_n^3 - 3x_n - 2}{3} \tag{1}$$

(i) If $x_i > 2$, $x_i^3 - 3x_i - 2 = x_i(x_i^2 - 3) - 2 > 2(4-3) - 2 = 0$. So, $x_{i+1} - x_i > 0$. Therefore, if $x_1 > 2$, the sequence is increasing and cannot converge to 2 or -1, so it must diverge.

(ii) If $x_i = 2$, $x_{i+1} - x_i = 0$. Thus, if $x_1 = 2$ the sequence is constant at 2, and it converges to 2.

(iii) If $-1 < x_i < 2$, we make the following claims:

(a) **Claim:** If $-1 < x_i < 2$, then $x_{i+1} - x_i \leq 0$.
It suffices to show that $y^3 - 3y - 2 \leq 0$ for all $y < 2$. Consider the expression $t(6-t) - 9$, $t > 0$. If $t > 6$, it is clear that $t(6-t) - 9 < 0$. If $0 < t \leq 6$, t and $6 - t$ are non negative. From the AM-GM inequality, we have $9 \leq t(6-t) \implies t(6-t) - 9 \leq 0$. Thus, $t(t(6-t) - 9) \leq 0$ is true for all $t > 0$. Now, let $y = 2 - t$. Plugging $t = 2 - y$ into the above inequality, we get $y^3 - 3y - 2 \leq 0$ for all $y < 2$.

(b) **Claim:** If $-1 < x_i < 2$, then $-1 < x_{i+1} < 2$.

$$x_i < 2 \implies x_i^3 < 8 \implies x_i^3 - 2 < 6 \implies \frac{x_i^3 - 2}{3} = x_{i+1} < 2$$

$$-1 < x_i \implies -1 < x_i^3 \implies -3 < x_i^3 - 2 \implies -1 < \frac{x_i^3 - 2}{3} = x_{i+1}$$

Thus, if x_1 lies between -1 and 2, then (x_n) is bounded and monotone decreasing. By the monotone convergence theorem, the sequence must converge, and since it cannot converge to 2, it must converge to -1.

(iv) If $x_i = -1$, $x_{i+1} - x_i = 0$. Thus, if $x_1 = -1$ the resulting sequence is constant at -1, and it converges to -1.

(v) For $x_i < -1$, we have $x_i^3 - 3x_i - 2 \leq 0$ from (iii)(a) (while claimed for $-1 < x_i < 2$, we proved it for all $x_i < 2$). Thus, for $x_1 < -1$ the sequence is decreasing. Since the sequence cannot converge to either -1 or 2, it must diverge to negative infinity.

2

Definitions
$\{(a_i, b_i)\}$ is a sequence in $\mathbb{R}^2$ with $a_i \leq b_i$ such that for each i, the closed interval $[a_i, b_i]$ contains the closed interval $[a_{i+1}, b_{i+1}]$.
$\sup a_n = \sup\{a_k \mid k \geq n\}$.

Solution

(a) **To prove:** $\bigcap_i [a_i, b_i] \neq \emptyset$.

Proof. Consider the sequences (a_n) and (b_n) independently. Since $[a_i, b_i]$ always contains $[a_{i+1}, b_{i+1}]$, (a_n) must be monotone increasing, and (b_n) must be monotone decreasing. Also, every term of (b_n) is an upper bound for (a_n), and every term of (a_n) is a lower bound for (b_n). So, $\alpha = \sup a_n$ and $\beta = \inf b_n$ must exist.

Claim: $\alpha \leq \beta$.
FTSOC, assume $\beta < \alpha$. From the definition of supremum, there must exist an a_{i_1} such that $\beta < a_{i_1} < \alpha$. From the definition of infimum, there must exist a b_{i_2} such that $\beta < b_{i_2} < a_{i_1} < \alpha$. This contradicts the fact that every term of (b_n) is an upper bound for (a_n). $\Rightarrow\Leftarrow$

Thus, the interval $[\alpha, \beta]$ is always non empty.

Claim: $\bigcap_i [a_i, b_i] = [\alpha, \beta]$
Every element in $[\alpha, \beta]$ is an upper bound for (a_i), and a lower bound for (b_i). Thus, $[\alpha, \beta] \subset [a_i, b_i]$ for all i, i.e, $[\alpha, \beta] \subset \bigcap_i [a_i, b_i]$.
If $k \in \bigcap_i [a_i, b_i]$, k is an upper bound of (a_n) and k is a lower bound for (b_n). So, $k \geq \alpha$ and $k \leq \beta$. Thus, $k \in [\alpha, \beta]$.

Thus, $\bigcap_i [a_i, b_i] = [\alpha, \beta] \neq \emptyset$. □

(b) **To prove:** $|\bigcap_i [a_i, b_i]| = 1 \iff \lim_{n\to\infty}(a_n - b_n) = 0$.

Proof. We know that if a sequence is bounded and increasing, it must converge to the least upper bound of its range. Thus, $(a_n) \to \alpha$. Similarly, if a sequence is bounded and decreasing, it must converge to the greatest lower bound of its range. Thus, $(b_n) \to \beta$.

Thus, we can say

$$
\begin{aligned}
&|\bigcap_i [a_i, b_i]| = 1 \\
\iff &\alpha = \beta \\
\iff &\lim_{n\to\infty}(a_n - b_n) = 0
\end{aligned}
$$

□

3

Definitions
(a_n) and (b_n) are sequences in $\mathbb{R}$.
$\sup a_n = \sup\{a_k \mid k \geq n\}$.

Solution

Lemma 3.1. $\sup(a_n + b_n) \leq \sup a_n + \sup b_n$ *for all* $n \in \mathbb{N}$

Proof. If $\sup(a_n)$ or $\sup(b_n)$ is ∞, the inequality is trivially true. Thus, assume $\sup(a_n)$, $\sup(b_n) \in \mathbb{R}$.
FTSOC, assume $\sup a_n + \sup b_n < \sup(a_n + b_n)$. There must exist $a_i + b_i,\ i \geq n$ such that $\sup a_n + \sup b_n < a_i + b_i < \sup(a_n + b_n)$. We know that $\sup a_n \geq a_k$ for all $k \geq n$. So, we have

$$
\begin{aligned}
&(\sup a_n - a_i) + \sup b_n < b_i \\
\implies &\sup b_n < b_i,
\end{aligned}
$$

which is impossible. $\Rightarrow\Leftarrow$

□

Lemma 3.2. *If* $(a_n) \to a \in \mathbb{R} \cup \{\infty, -\infty\}$*,* $(b_n) \to b \in \mathbb{R} \cup \{\infty, -\infty\}$*, and* $a_n \leq b_n$ *for all* $n \in \mathbb{N}$*, then* $a \leq b$*.*

Proof. If $(b_n) \to \infty$, the inequality is trivially true. If $(a_n) \to \infty$, it must be the case that $(b_n) \to \infty$. Thus, assume $a, b \in \mathbb{R}$.
FTSOC, assume $b < a$. Let $\epsilon = \frac{a-b}{2}$. Then, there exists an integer N_1 such that for all $n \geq N_1$, $|a_n - a| < \epsilon$. Similarly, there exists an integer N_2 such that for all $n \geq N_2$, $|b_n - b| < \epsilon$. Let N be the greater of N_1 and N_2. Thus, for all $n \geq N$, $b_n \in \left(\frac{3b-a}{2}, \frac{a+b}{2}\right)$ and $a_n \in \left(\frac{a+b}{2}, \frac{3a-b}{2}\right)$, which implies $a_n > b_n$ for all $n \geq N$. $\Rightarrow\Leftarrow$

□

To prove: $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, provided the sum on the right is not of the form $\infty - \infty$.

Proof. Consider the sequences $(\sup a_n + \sup b_n)$, and $(\sup(a_n + b_n))$. From Lemma 3.1, we have $\sup(a_n + b_n) \leq \sup a_n + \sup b_n$ for all $n \in \mathbb{N}$. Also, it is clear the the suprema of tails of a sequence form a monotone decreasing sequence. Since all sequences are bounded in the extended reals, it follows these sequences must converge to a value in the extended reals. Thus, we can use Lemma 3.2:

$$\begin{aligned} & \sup(a_n + b_n) \leq \sup a_n + \sup b_n \\ \implies & \lim_{n\to\infty}(\sup(a_n + b_n)) \leq \lim_{n\to\infty}(\sup a_n + \sup b_n) = \lim_{n\to\infty}(\sup a_n) + \lim_{n\to\infty}(\sup b_n) \\ \implies & \limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \end{aligned}$$

□

4

Definitions
(p_n) and (q_n) are Cauchy sequences in a metric space (X, d).

Solution

To prove: The sequence $(d(p_n, q_n))$ converges.

Proof. For any m, n, from the triangle inequality, we have

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n)$$

and

$$d(p_m, q_n) \leq d(p_m, q_m) + d(q_m, q_n).$$

Combining the two, we get

$$\begin{aligned} & d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \\ \implies & |d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n). \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Since (p_n) and (q_n) are Cauchy sequences, there must exist integers N_1 and N_2 such that for all $m, n \geq N_1$, $d(p_n, p_m) < \frac{\epsilon}{2}$ and for all $m, n \geq N_2$, $d(q_n, q_m) < \frac{\epsilon}{2}$ respectively. Let N be the larger of N_1 and N_2. Thus,

$$|d(p_n, q_n) - d(p_m, q_m)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq N$. This implies that the sequence $(d(p_n, q_n))$ is Cauchy. In $\mathbb{R}$, a sequence is Cauchy if and only if it is convergent (to a limit in $\mathbb{R}$). □

5

Definitions
(p_n) is a sequence in $\mathbb{R}$.
For $a < b$, let $\zeta_{a,b}$ be the $[a,b]$-crossing number of (p_n).

Solution

To prove: $\zeta_{a,b} \in \mathbb{N}_0 \;\; \forall\, a < b \iff (p_n) \to p \in \mathbb{R} \cup \{\infty, -\infty\}$.

Proof. We will call a sequence with the property on the left (having finite $\zeta_{a,b}$ for all $a < b$) a "cross-bounded" sequence.

Proof of convergent $\implies$ cross-bounded

We will prove the contrapositive: not cross-bounded $\implies$ not convergent. If a sequence is not cross-bounded, $\zeta_{a,b} = \infty$ for some $a < b$. This implies that for all N, there exists $d_n > N$ such that $p_{d_n} < a$, and there exists $u_n > N$ such that $p_{u_n} > b$. This allows us to make the following deductions:

(i) For $\epsilon = b - a$, there does not exist an N such that for all $m, n \geq N$, $|p_m - p_n| < \epsilon$. This tells us that (p_n) is not Cauchy, which implies (p_n) is not convergent in $\mathbb{R}$.

(ii) For all $M > a$, there does not exist an N such that for all $n \geq N$, $p_n \geq M$. Thus, (p_n) cannot converge to ∞.

(iii) For all $m < b$, there does not exist an N such that for all $n \geq N$, $p_n \leq m$. Thus, (p_n) cannot converge to $-\infty$.

Proof of cross-bounded $\implies$ convergent

We can rephrase the notion of being cross-bounded like so: for every $a < b$, there either exists an N such that for all $n \geq N$, $p_n \geq a$, or there exists an N such that for all $n \geq N$, $p_n \leq b$. In other words, eventually a becomes a lower bound for tails of (p_n), or b becomes an upper bound for tails of (p_n). We will analyze bounded (in $\mathbb{R}$) and unbounded cases for (p_n) separately.

(i) Say (p_n) is bounded in $\mathbb{R}$, i.e, $m < p_n < M$ for all $n \in \mathbb{N}$ for some $m, M \in \mathbb{R}$. Consider the interval

$$(\alpha, \beta) := \left(\frac{3m + M}{4}, \frac{m + 3M}{4} \right).$$

Given our paraphrasing of what it means to be cross-bounded, there either exists an N such that for all $n \geq N$, $p_n \geq \alpha$, or there exists an N such that for all $n \geq N$, $p_n \leq \beta$. In both cases, we have achieved new bounds for (p_n) for $n \geq N$, namely (α, M) and (m, β) respectively. Notice that $M - a = \beta - m = \frac{3}{4}(M - m)$. Effectively, we have bounded a tail of the sequence in an interval that is three-fourths the size of the interval we started with. If we repeat this process k times (always taking the maximum of all the N's we've chosen), we will bound a tail of (p_n) in an interval of the size $\left(\frac{3}{4}\right)^k (M-m)$. Taking the limit as $k \to \infty$,

$$\lim_{k\to\infty} (M-m)\left(\frac{3}{4}\right)^k = (M-m)\lim_{k\to\infty}\left(\frac{3}{4}\right)^k = 0$$

Thus, for arbitrary $\epsilon > 0$, there will exist k such that $(M-m)\left(\frac{3}{4}\right)^k < \epsilon$, i.e, there will exist a tail of (p_n) where the difference between any two terms is less than ϵ. This implies that (p_n) is Cauchy, and hence, convergent.

(ii) If (p_n) is not bounded in $\mathbb{R}$, we will show that it converges to $+\infty$ or $-\infty$. Consider any interval (α, β). As stated previously, one of these should be true:

(a) There exists an N such that for all $n \geq N$, $p_n \geq \alpha$.

Since (p_n) is unbounded in $\mathbb{R}$ by hypothesis, and α functions as a lower bound for all $n \geq N$, (p_n) cannot have an upper bound. Thus, for any interval (α', β'), where $\alpha < \alpha' < \beta'$, there must exist an N' such that for all $n \geq N'$, $p_n \geq \alpha'$. Since α' was arbitrary, this shows that $(p_n) \to \infty$.

(b) There exists an N such that for all $n \geq N$, $p_n \leq \beta$.

Since (p_n) is unbounded in $\mathbb{R}$ by hypothesis, and β functions as an upper bound for all $n \geq N$, (p_n) cannot have an lower bound. Thus, for any interval (α', β'), where $\alpha' < \beta' < \beta$, there must exist an N' such that for all $n \geq N'$, $p_n \leq \beta'$. Since β' was arbitrary, this shows that $(p_n) \to -\infty$.

$\square$