Analysis 1 HW 1

Exercise 1

Definitions

$f : S \to T$ is invertible $\iff \exists\, g : T \to S$ with $fg = id_T$ and $gf = id_S$.

$f : S \to T$ is injective $\iff f(x_1) = f(x_2) \implies x_1 = x_2 \;\; \forall\, x_1,\, x_2 \in S$.

$f : S \to T$ is surjective $\iff \forall\, y \in T \;\; \exists\, x \in S$ with $f(x) = y$.

(a) **P:** f is invertible.

Q: f is injective and f is surjective.

To prove $P \iff Q$, we need to prove $P \implies Q$ and $\neg P \implies \neg Q$.

(i) $P \implies Q$

Proof. From P, we have $g(f(x)) = x \;\; \forall\, x \in S$. Consider $x_1, x_2 \in S$. Then,

$$
\begin{aligned}
g(f(x_1)) &= x_1 \\
g(f(x_2)) &= x_2
\end{aligned}
$$

Since g is a function, $f(x_1) = f(x_2) \implies x_1 = x_2$.
$\therefore f$ is injective.

Also from P, $f(g(y)) = y \;\; \forall\, y \in T$. For this to be true, every element in T must be in the image of f.
$\therefore f$ is surjective.

$\square$

(ii) $\neg P \implies \neg Q$

Proof.

$$
\begin{aligned}
\neg P &\iff \nexists\, g : T \to S \text{ with } fg = id_T \text{ and } gf = id_S \\
&\iff \forall\, g : T \to S \;\; fg \neq id_T \text{ or } gf \neq id_S \\
&\iff (\exists\, y \in T \text{ with } y \notin f(S)) \text{ or} \\
&\qquad (\exists\; x_1, x_2 \in S,\;\; x_1 \neq x_2,\;\; \text{with } f(x_1) = f(x_2) = y,\;\; y \in T) \\
&\iff (f \text{ is not surjective}) \text{ or } (f \text{ is not injective}) \\
&\iff \neg Q
\end{aligned}
$$

$\square$

(b) Let g and h be inverses of f. Then,

$$f(g(x)) = x \quad \forall\, x \in T$$
$$f(h(x)) = x \quad \forall\, x \in T$$

$$\implies fg = fh$$

If f is invertible, from (a), f is injective. $\therefore\, g = h$, i.e, the inverse of an invertible function is unique.

(c) (i) **Left invertible functions:** From (a)(i), $f : S \to T$ being left invertible is equivalent to f being injective. Let $g : T \to S$ be a left inverse of f. While f being injective guarantees that g exists, g will be unique if and only if f is also surjective, as otherwise $g(y),\ y \in T \setminus f(S)$ can be arbitrarily defined as any $x \in S$.

(ii) **Right invertible functions:** From (a)(i), $f : S \to T$ being right invertible implies f is surjective. Let $g : T \to S$ be a right inverse of f. g is unique if and only if f is also injective, as otherwise $g(y),\ y \in T$ can be arbitrarily defined to be any $x \in S$ such that $f(x) = y$.

(iii) **Left and right invertible functions:** If f is left invertible and right invertible, it is injective and surjective, which implies f is invertible, which implies f has a unique inverse, which must be equal to its left and right inverses.

(iv) **Left cancellable functions:** Let $h_1, h_2 : T \to S$. For $f : S \to T$ to be left cancellable,

$$\begin{aligned} &(fh_1 = fh_2) \implies (h_1 = h_2) \\ \iff\ &(f(x_1) = f(x_2)) \implies (x_1 = x_2) \ \ \forall\, x_1, x_2 \in S \end{aligned}$$

which implies f is injective. Evidently, the terms "injective", "left invertible", and "left cancellable" are equivalent.

(v) **Right cancellable functions:** Borrowing h_1, h_2, and f from (iv), f is right cancellable if

$$\begin{aligned} &(h_1 f = h_2 f) \implies (h_1 = h_2) \\ \iff\ &(h_1 f(x) = h_2 f(x) \ \ \forall\, x \in S) \implies (h_1(y) = h_2(y) \ \ \forall\, y \in T) \\ \iff\ &\forall\, y \in T \ \ \exists\, x \in X \text{ with } f(x) = y \end{aligned}$$

which implies f is surjective. Evidently, the terms "surjective", "right invertible", and "right cancellable" are equivalent.

(vi) **Left and right cancellable functions:** If f is left cancellable and right cancellable, it is injective and surjective, which implies f is invertible.

(d) The claim is no longer true if the condition of the domain and codomain being of finite cardinality is dropped. For example, $f : \mathbb{N} \to \mathbb{R}$ defined as $f(x) = x$ is injective, and has a domain and a codomain both of infinite cardinality. However, one is countably infinite which the other is uncountably infinite, making it impossible for the function to be surjective.

Exercise 2

Definitions
S is the universal set.
A is the set of all equivalence relations on S.
B is the set of all partitions of S.
$[s]_a = \{k \in S \mid (k, s) \in a\}, s \in S, a \in A$ is the equivalence class of s in a.
$f : A \to B,\ f(a) = \{[s]_a \mid s \in S\}$ maps equivalence relations to partitions.
$g : B \to A,\ g(b) = \{(n, m) \mid \exists\ e \in b \text{ with } n, m \in e\}$ maps partitions to equivalence classes.

Claim
f and g are inverses of each other.

(i) $f(a), a \in A$ is a partition.

Proof. (a) For $s_1, s_2 \in S, a \in A$ if $[s_1]_a \cap [s_2]_a \neq \phi$, there exists x such that $x \in [s_1]_a$ and $x \in [s_2]_a$. This implies $(x, s_1) \in a$ and $(x, s_2) \in a$. Since a is symmetric, $(s_1, x) \in a$, which implies $(s_1, s_2) \in a$. Thus, $[s_1]_a = [s_2]_a$. $\therefore$ the elements of $f(a)$ are disjoint.

(b) For every equivalence relation $a \in A$, $(s, s) \in a \quad \forall s \in S$. Thus, there exists $e \in f(a)$ such that $s \in e, \quad \forall s \in S$. $\therefore$ $f(a)$ is exhaustive.

$f(a)$ is disjoint and exhaustive, which implies that $f(a)$ is a partition of S. □

(ii) $g(b), b \in B$ is an equivalence relation.

Proof. (a) $\exists e \in b$ such that $s \in e \quad \forall s \in S$. Thus, $(s, s) \in g(b) \quad \forall s \in S$. $\therefore$ $g(b)$ is reflexive.

(b) $s_1, s_2 \in e \iff s_2, s_1 \in e$. Thus, if $(s_1, s_2) \in g(b)$, $(s_2, s_1) \in g(b) \quad \forall s_1, s_2 \in S$. $\therefore$ $g(b)$ is symmetric.

(c) $s_1, s_2 \in e_1$ and $s_2, s_3 \in e_2$ for some $e_1, e_2 \in b$, $s_2 \in e_1 \cap e_2$, which implies that $e_1 = e_2$. Hence, If $(s_1, s_2) \in g(b)$ and $(s_2, s_3) \in g(b)$, $(s_1, s_3) \in g(b)$. $\therefore$ $g(b)$ is transitive.

$g(b)$ is reflexive, symmetric, and transitive, which implies $g(b)$ is an equivalence relation. □

(iii) $fg = id_B$

Proof. Let $s_1, s_2 \in S$. Let $b \in B$, so b is a partition of S. From the definition of g, $(s_1, s_2) \in g(b)$ if and only if s_1 and s_2 belong to the same part of b. From the definition of f, $\exists e \in f(g(b))$ such that $s_1, s_2 \in e$ if and only if $(s_1, s_2) \in g(b)$. Thus,

$$\begin{aligned} &\exists e \in b \text{ such that } s_1, s_2 \in e \\ \iff &(s_1, s_2) \in g(b) \\ \iff &\exists e \in f(g(b)) \text{ such that } s_1, s_2 \in e \end{aligned}$$

$\therefore f(g(b)) = b \ \ \forall b \in B$. □

(iv) $gf = id_A$

Proof. Let $s_1, s_2 \in S$. Let $a \in A$. From the definition of f, s_1 and s_2 belong to the same part of $f(a)$ if and only if $(s_1, s_2) \in a$. From the definition of g, $(s_1, s_2) \in g(f(a))$ if and only if s_1 and s_2 belong to the same part of $f(a)$. Thus,

$$\begin{aligned} & (s_1, s_2) \in a \\ \iff & \exists e \in f(a) \text{ such that } s_1, s_2 \in e \\ \iff & (s_1, s_2) \in g(f(a)) \end{aligned}$$

$\therefore g(f(a)) = a \ \ \forall a \in A.$ $\square$

From (iii) and (iv), f is bijective. This implies that there exists a one to one correspondence between the elements of A and those of B. Thus, defining a partition of S is "the same as" defining a equivalence relation on S in the sense that every partition corresponds uniquely to an equivalence relation and vice versa.

Exercise 3

Definitions
$f : S \to T$
$f(A) = \{f(x) \mid x \in A\}$, $A \subset S$
$f^{-1}(B) = \{x \in S \mid f(x) \in B\}$, $B \subset T$

(a) $f(\bigcup_{i\in I} A_i) = \bigcup_{i \in I} f(A_i)$, $A_i \subset S$

Proof.

$$\begin{aligned} & y \in f\Big(\bigcup_{i\in I} A_i\Big) \\ \iff & \exists A_i \text{ such that } \exists x \in A_i \text{ such that } f(x) = y \\ \iff & \exists A_i \text{ such that } y \in f(A_i) \\ \iff & y \in \bigcup_{i\in I} f(A_i) \end{aligned}$$

$\square$

(b) $f(\bigcap_{i\in I} A_i) \neq \bigcap_{i\in I} f(A_i)$ $A_i \subset S$

Proof. Consider an x_i in every A_i, such that no x_i is in $\bigcap_{i\in I} A_i$. Let $f(x_i) = y \ \ \forall\, i \in I$, and $f(x) \neq y$ for all other $x \in S$. Clearly, $y \notin f(\bigcap_{i\in I} A_i)$. However, $y \in f(A_i) \ \ \forall\, i \in I \implies y \in \bigcap_{i\in I} f(A_i)$. $\square$

(c) $f(A^c) \neq f(A)^c$

Proof. Consider $x_1, x_2 \in S$ such that $x_1 \in A$ and $x_2 \notin A$. Let $f(x_1) = f(x_2) = y$. Now, $x_2 \in A^c \implies y \in f(A^c)$. However, $x_1 \in A \implies y \in f(A) \implies y \notin f(A)^c$. $\square$

(d) (i) $f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i)$, $B_i \subset T$

Proof.

$$\begin{aligned} & x \in f^{-1}\Big(\bigcup_{i\in I} B_i\Big) \\ \iff & \exists B_i \text{ such that } f(x) \in B_i \\ \iff & \exists B_i \text{ such that } x \in f^{-1}(B_i) \\ \iff & x \in \bigcup_{i\in I} f^{-1}(B_i) \end{aligned}$$

$\square$

(ii) $f^{-1}(\bigcap_{i\in I} B_i) = \bigcap_{i\in I} f^{-1}(B_i)$ $B_i \subset T$

Proof.

$$\begin{aligned} & x \in f^{-1}\Big(\bigcap_{i\in I} B_i\Big) \\ \iff & f(x) \in B_i \quad \forall\, i \in I \\ \iff & x \in f^{-1}(B_i) \quad \forall\, i \in I \\ \iff & x \in \bigcap_{i\in I} f^{-1}(B_i) \end{aligned}$$

$\square$

(iii) $f^{-1}(B^c) = f^{-1}(B)^c$

Proof.

$$\begin{aligned} & x \in f^{-1}(B^c) \\ \iff & f(x) \in B^c \\ \iff & f(x) \notin B \\ \iff & x \notin f^{-1}(B) \\ \iff & x \in f^{-1}(B)^c \end{aligned}$$

$\square$

(e) (i) $f^{-1}(f(A)) \neq A$.

Proof. Consider $x_1 \in A,\ x_2, \notin A,\ f(x_1) = f(x_2) = y$. $y \in f(A)$, which implies $x_1, x_2 \in f^{-1}(f(A))$. $\square$

(ii) $f(f^{-1}(B)) \neq B$.

Proof. Consider $y \in T$ such that y is not in the image of f. Consider $B \subset T$ such that $y \in B$. Since there does not exist x in $f^{-1}(B)$ such that $f(x) = y$, $y \notin f(f^{-1}(B))$ □

(f) Parts (b), (c), and (e)(i) relied on f not being injective to provide counterexamples. If f is injective, it can be proved that f preserves intersections and complements, like (d)(ii) and (d)(iii). In (e)(ii), $f(f^{-1}(B)) = B$ if f is surjective.