Analysis 1 Quiz 0 Answers
Comments are welcome

1. (*Entrance exam question with a minor variation.*) Suppose a differentiable function f from $\mathbb{R}$ to $\mathbb{R}$ has a local *minimum* at $(a, f(a))$. This means there are real numbers m and M such that (i) $m < a < M$ and (ii) $f(a) \leq f(x)$ for any $x \in [m, M]$. The proof of a standard result is sketched below. Complete it as instructed using the given options.

Proof: For sufficiently large $h < 0$ (i.e., sufficiently small $|h|$ with $h < 0$), it is given that $f(a+h) \geq f(a)$.

Therefore for such h the quantity $\frac{f(a+h)-f(a)}{h}$ must be ≤ 0.

By taking the limit of this quantity as $h \to 0$ from the appropriate side, we get that $f'(a)$ must be ≤ 0.

A parallel argument for suitable positive values of h gives that $f'(a)$ must be ≥ 0.

Combining both conclusions gives the desired result: $f'(a) = 0$. Note that the mentioned limits exist because f is differentiable.

Options

A. small	B. large	C. $\geq$
D. $>$	E. $\leq$	F. $<$
G. $=$	H. $\neq$	I. 0
J. $f(a)$	K. $\frac{f(a+h)-f(a)}{h}$	L. $f'(a)$
M. f is differentiable	N. f is continuous	

2. Consider the following calculation, where L'Hôpital's rule is used in the first step. Note that as $x \to 0$, values of $\cos(x^{-1})$ and $\sin(x^{-1})$ keep oscillating but stay bounded between -1 and 1.

$$\lim_{x\to 0} \frac{x^2 \sin(x^{-1})}{\sin x} = \lim_{x\to 0} \frac{2x\sin(x^{-1}) + (x^2)(-x^{-2})\cos(x^{-1})}{\cos x} = \lim_{x\to 0} \frac{2x\sin(x^{-1})}{\cos x} - \lim_{x\to 0} \frac{\cos(x^{-1})}{\cos x}.$$

Of the two limits in the last step, the first is 0 due to the factor $2x$ but the second does not exist because $\cos(x^{-1})$ keeps oscillating in $[-1, 1]$ as $x \to 0$. So the original limit does not exist. **Is this reasoning right?**

Answer: No, the very first step is unjustified. For the theorem underlying L'Hôpital's rule (namely $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ under suitable conditions), the existence of $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ is a hypothesis, as we will see in Rudin's theorem 5.13. In this problem $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ does not exist, as correctly deduced above. So L'Hôpital's rule is inapplicable and we cannot use it to draw any conclusion about the original limit $\lim_{x\to 0} \frac{x^2\sin(x^{-1})}{\sin x}$. However, $\lim_{x\to 0} \frac{x^2}{\sin x} = 0$, so the desired limit is also 0 by the squeeze theorem .

Turn over →

3. (*Entrance exam question expanded.*) Suppose f is a function whose domain is X and codomain is Y. It is given that $|X| > 1$ and $|Y| > 1$. No other information is known about X, Y and f. For each statement listed below, write the numbers of *all* correct options (and no incorrect options!) that apply to that statement.

Statements and answers: Only the strongest conclusions are listed.

a) For each x in X and for each y in Y it is true that $f(x) = y$.

Answer: False (unless $|X| = |Y| = 1$, or one of the sets is empty — which is not the case here)

b) For each x in X, there exists y in Y such that $f(x) = y$. Answer: True (part of definition of a function)

c) For each y in Y, there exists x in X such that $f(x) = y$. Answer: Equivalent to onto

d) There exists x in X and there exists y in Y such that $f(x) = y$.

Answer: True (because both sets are nonempty)

e) For each x in X, there exists a unique y in Y such that $f(x) = y$. Answer: True (definition of a function)

f) For each y in Y, there exists a unique x in X such that $f(x) = y$.

Answer: Equivalent to f being a bijection (i.e., both one-to-one and onto)

g) There exists a unique x in X and there exists a unique y in Y such that $f(x) = y$.

Answer: False (unless $|X| = 1$, which is not the case here)

h) There exists a unique x in X such that for each y in Y it is true that $f(x) = y$.

Answer: False (unless $|X| = |Y| = 1$, which is not the case here)

i) There exists a unique y in Y such that for all x in X it is true that $f(x) = y$.

Answer: Equivalent to f being a constant function (by definition)

Note: in the next two statements, the symbol $\forall$ stands for "for all"

j) $\forall\, x_1$ in X and $\forall\, x_2$ in X and $\forall\, y$ in Y, if $f(x_1) = f(x_2) = y$ then $x_1 = x_2$.

Answer: Equivalent to one-to-one

k) $\forall\, y_1$ in Y and $\forall\, y_2$ in Y, and $\forall\, x$ in X, if $f(x) = y_1 = y_2$ then $y_1 = y_2$.

Answer: True (part of definition of a function)

Options

1. The statement is true.
2. The statement is false.
3. If the statement is true then f is one-to-one.
4. If f is one-to-one then the statement is true.
5. If the statement is true then f is onto.
6. If f is onto then the statement is true.
7. If the statement is true then f is constant.
8. If f is constant then the statement is true.
9. None of the above.