HOMEWORK 5 - ALGEBRA I - AUGUST-NOVEMBER 2024

(1) In this question, the matrices are defined over complex numbers. $n > 1$ is an integer.

- (i) Show that a polynomial of degree n defined over any field F has at most n roots.
- (ii) Show that the set of $n \times n$ matrices commuting with a fixed $n \times n$ matrix A is a vector subspace of $\mathbb{C}^n$. Compute the dimension of this space.
- (iii) Show that if A is an $n \times n$ matrix which has n distinct eigenvalues, and B is an $n \times n$ matrix such that A and B commute, then B is a polynomial of degree at most $n-1$ in the matrix A. That is, there exists a polynomial $p(x)$ with complex coefficients and degree at most $n-1$ such that $p(A) = B$.

(2) Show that a bilinear form $\langle , \rangle$ on a real vector space V is a sum of a symmetric form and a skew-symmetric form.

(3) Let A and A' be real symmetric matrices such that $A' = P^t A P$, where P is an invertible matrix. Are ranks of A and A' equal?

(4) Show that the set of $n \times n$ Hermitian matrices forms a real vector space and compute its dimension.

(5) Show the following if true, or demonstrate that it is false by finding a counterexample and salvage the result if possible by adding a suitable hypothesis or weakening the conclusion :

- (i) If A is a real $m \times n$ matrix, $B = A^t A$ is a symmetric positive definite matrix of the same rank as A.
- (ii) If A is a complex $m \times n$ matrix, $B = A^* A$ is a unitary positive definite matrix of the same rank as A.

(6) Show that the signature of a bilinear form $\langle , \rangle$ is independent of the orthogonal basis chosen to write its matrix.

(7) Let $V = \mathbb{R}_{2\times 2}$ be the real vector space of real 2×2 matrices. Let $W = \mathbb{C}_{2\times 2}$ be the real vector space of complex 2×2 matrices.

- (i) Show that the function $\langle , \rangle : V \times V \to \mathbb{R}$ given by $\langle A, B \rangle = \mathrm{Trace}(AB)$ is a symmetric bilinear form. Determine its matrix with respect to the standard basis. Determine its signature.
- (ii) Show that the function $\langle , \rangle : V \times V \to \mathbb{R}$ given by $\langle A, B \rangle = \mathrm{Trace}(A^t B)$ is a symmetric bilinear form. Determine its matrix with respect to the standard basis. Determine its signature.
- (iii) Is the function $\langle , \rangle : W \times W \to \mathbb{C}$ given by $\langle A, B \rangle = \mathrm{Trace}(A^* B)$ a Hermitian form? If so, determine its signature.
- (iv) Is the function $\langle , \rangle : W \times W \to \mathbb{C}$ given by $\langle A, B \rangle = \mathrm{Trace}(\bar{A} B)$ a Hermitian form? (The $\bar{A}$ denotes the complex conjugation applied to all entries of the matrix A.) If so, determine its signature.

(8) Let V be the real vector space of 3×3 matrices equipped with the bilinear form $\langle A, B \rangle = \mathrm{Trace}(A^t B)$ and let W denote the subspace of skew-symmetric matrices. Compute the orthogonal projection to W of the matrix

$$M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

with respect to this form.

(9) For a Euclidean space V, prove the parallelogram law:

$$\|v + w\|^2 + \|v - w\}^2 = 2\|v\|^2 + 2\|w\|^2.$$

Does it hold in a Hermitian space?

(10) Let $w \in \mathbb{R}^n$ be a vector of length 1 under the usual dot product. Recall that for any vector v, we can uniquely write $v = cw + u$ where $u \in W^\perp$ and c is a real number. Define $r_w(v) := -cw + u$.

(i) Prove that the matrix $P = I - 2ww^t$ is orthogonal, i.e. $PP^t = I$.

(ii) Prove that multiplication by P is a reflection about the space $W^\perp$. Is this the same map as r_w?

(iii) Show that the linear operator given by P preserves lengths of vectors.

(iv) If v, v' are two vectors of equal length, find a vector w such that $Pv = v'$.

(11) Let $T : V \to V$ be a linear transformation on $V = \mathbb{R}^n$ such that its matrix in some basis is a real symmetric matrix. Prove that $V = \ker(T) \oplus \operatorname{im}(T)$.

(12) Let T be a unitary operator on a Hermitian space $(V, \langle , \rangle)$ and let v_1, v_2 be two eigenvectors of T with distinct eigenvalues. Show that v_1 and v_2 are orthogonal.