HOMEWORK 4 - ALGEBRA I - AUGUST-NOVEMBER 2024

(1) Recall that a linear operator T is *nilpotent* if some positive power T^k is zero. Prove that T is nilpotent if and only if there is a basis of V such that the matrix of T is upper triangular, with diagonal entries zero.

(2) Let

$$M = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

be a matrix in block form. Show that M is diagonalizable if and only if A and D are diagonalizable.

(3) Let M be a 2×2 matrix with eigenvalue λ.

(i) Show that unless it is zero, the vector $(b, \lambda - a)^t$ is an eigenvector.

(ii) Find a matrix P such that $P^{-1}MP$ is diagonal, assuming that $b \neq 0$ and that M has distinct eigenvalues.

(4) Let M be the $n \times n$ matrix given as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

(i) Show that $M^n = 0$ but $M^{n-1} \neq 0$.

(ii) Find all $n \times n$ matrices N over complex numbers that commute with M, i.e. $MN = NM$.

(5) Consider the complex numbers $\mathbb{C}$ as a 2-dimensional vector space V over the real numbers $\mathbb{R}$. For any complex number α, consider the function $T_\alpha : V \to V$ given by $v \to \alpha v$ obtained by simply multiplying by α. Using the basis $1, \iota$ of complex numbers over real numbers, find matrix of T_α. Show that the set $\{T_\alpha\}$ satisfies all properties of complex numbers that you know.

(6) If A and B are two square matrices which are both diagonalizable such that $AB = BA$, then show that there exists an invertible matrix C such that CAC^{-1} and CBC^{-1} are both diagonal.

(7) Let V be a vector space over a field F and let W be a subspace. Define

$$A(W) := \{f \in \hat{V} : f(w) = 0 \text{ for all } w \in W\}.$$

(i) Show that $A(W)$ is a subspace of $\hat{V}$ and that if $W \subset U$, then $A(U) \subset A(W)$.

(ii) Show that dimension of $A(W)$ equals $\dim V - \dim W$.

(iii) Show that $A(A(W)) = W$.

(8) Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Define distance $d(u, v) := \|u - v\|$. Show that it satisfies the triangle inequality.

(9) If $\{w_1, w_2, \ldots, w_n\}$ is an orthornomal set in an inner product space V, show that

$$\sum_i |\langle w_i, v\rangle|^2 \leq \|v\|^2$$

for any $v \in V$.