HOMEWORK 3 - ALGEBRA I - AUGUST-NOVEMBER 2024

(1) A matrix B is *symmetric* if $B = B^t$, where ^{t} denotes the transpose of the matrix.

(i) Show that for any square matrix B, $B + B^t$ and BB^t are symmetric.

(ii) If A is invertible, then $(A^{-1})^t = (A^t)^{-1}$.

(iii) Let A and B be symmetric $n \times n$ matrices. Prove that the product AB is symmetric if and only if $AB = BA$.

(2) This exercise is about determinants of block matrices.

(i) Show that the determinant of the matrix

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

is computed as $\det M = (\det A)(\det D)$, if A and D are square blocks.

(ii) Let a $2n \times 2n$ matrix M be given in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Suppose that A is invertible and that $AC = CA$. Use block multiplication to prove that $\det M = \det(AD - CB)$. Give an example to show that this formula need not hold if $AC \neq CA$. Does the formula hold if A is not invertible but $AC = CA$?

(3) Determine the smallest integer n such that every invertible 2×2 matrix can be written as a product of at most n elementary matrices.

(4) Let $x_1, x_2, \ldots, x_n$ be variables. Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}.$$

(5) Let V denote the space $\mathbb{R}_{n\times n}$ of all $n \times n$ real matrices.

(i) V is the direct sum of the space of symmetric matrices and the space of *skew-symmetric* matrices, where skew-symmetric matrices are defined as the matrices A satisfying $A^t = -A$.

(ii) Let W denote the subspace of V of matrices with trace zero. Find $W' \subset V$ a subspace such that $V = W \oplus W'$.

(6) Let V be a vector space over an infinite field F. Prove that V is not the union of finitely many proper subspaces.