HOMEWORK 2 - ALGEBRA I - AUGUST-NOVEMBER 2024

All matrices and objects that appear in this homework are defined over the real numbers.

(1) Give an example of a 6×7 matrix A with dimension of nullspace of $A = 3$, or give an argument why such a matrix cannot exist.

(2) Give an example of a 5×8 matrix A with dimension of nullspace of $A = 2$, or give an argument why such a matrix cannot exist.

(3) Find a basis for the subspace of all vectors in $\mathbb{R}^5$ satisfying

$$x_1 + 3x_2 + 2x_3 - x_4 - 7x_5 = 0.$$

(4) Find a basis for the image and the kernel of the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(5) For any matrix A, let A^T denote its transpose matrix. Verify the following claim: For A any 2×3 matrix, a vector is in the kernel of A^T if and only if it is perpendicular (in the geometric sense) to the image of A.

(6) Recall that the *trace* of a square matrix is defined to be the sum of its diagonal entries.

- Show that trace is a linear map from the vector space of square matrices to the vector space $\mathbb{R}^1$.
- Show that $\text{trace}(AB) = \text{trace}(BA)$.
- Show that if B is invertible, then $\text{trace}(A) = \text{trace}(BAB^{-1})$.
- Show that the equation

$$AB - BA = \begin{bmatrix} a_0 & & & & \\ & a_1\pi & & & \\ & & a_2\pi^2 & & \\ & & & \ddots & \\ & & & & a_{n-1}\pi^{n-1} \end{bmatrix}$$

  where π is the well-known transcendental ratio of the circumeference of a circle with its diameter and $a_0, \dots, a_{n-1}$ are arbitrary rational numbers, has no solution in $n \times n$ matrices A, B for any n.

(7) Let X be a finite set. Consider the space $\mathcal{F}(X, \mathbb{R})$ to be the set of functions $f : X \to \mathbb{R}$. Show that this is a vector space under pointwise addition and scalar multiplication. What is the dimension of this space if X is finite?

(8) Classify all the linear transformations T in the following cases:

- $T : \mathbb{R} \to \mathbb{R}$.
- $T : \mathbb{R} \to \mathbb{R}^2$.
- $T : \mathbb{R}^2 \to \mathbb{R}$.
- $T : \mathbb{R}^2 \to \mathbb{R}^2$.

What does this say geometrically?