HOMEWORK 2 - ALGEBRA I - AUGUST-NOVEMBER 2024

All matrices and objects that appear in this homework are defined over the real numbers.

(1) Give an example of a 6×7 matrix A with dimension of nullspace of A = 3, or give an argument why such a matrix cannot exist.

(2) Give an example of a 5×8 matrix A with dimension of nullspace of A = 2, or give an argument why such a matrix cannot exist.

(3) Find a basis for the subspace of all vectors in \mathbb{R}^5 satisfying

$$x_1 + 3x_2 + 2x_3 - x_4 - 7x_5 = 0$$

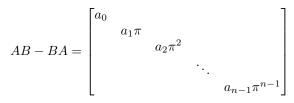
(4) Find a basis for the image and the kernel of the following matrix:

[1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1

(5) For any matrix A, let A^T denote its transpose matrix. Verify the following claim: For A any 2 × 3 matrix, a vector is in the kernel of A^T if and only if it is perpendicular (in the geometric sense) to the image of A.

(6) Recall that the *trace* of a square matrix is defined to be the sum of its diagonal entries.

- Show that trace is a linear map from the vector space of square matrices to the vector space \mathbb{R}^1 .
 - Show that trace(AB) = trace(BA).
 - Show that if B is invertible, then $trace(A) = trace(BAB^{-1})$.
 - Show that the equation



where π is the well-known transcendental ratio of the circumference of a circle with its diameter and a_0, \ldots, a_{n-1} are arbitrary rational numbers, has no solution in $n \times n$ matrices A, B for any n.

(7) Let X be a finite set. Consider the space $\mathcal{F}(X,\mathbb{R})$ to be the set of functions $f: X \to \mathbb{R}$. Show that this is a vector space under pointwise addition and scalar multiplication. What is the dimension of this space if X is finite?

(8) Classify all the linear transformations T in the following cases:

- $T : \mathbb{R} \to \mathbb{R}$.
- $T: \mathbb{R} \to \mathbb{R}^2$.
- $T: \mathbb{R}^2 \to \mathbb{R}$.
- $T: \mathbb{R}^2 \to \mathbb{R}^2$

What does this say geometrically?