HOMEWORK 1 - ALGEBRA I - AUGUST-NOVEMBER 2024

All matrices and objects that appear in this homework are defined over the real numbers.

(1) Define E_{ij} to be the matrix with 1 in row i and column j and 0 in every other entry. Write the product $E_{ij}E_{kl}$.

(2) Find the elementary matrices E_1, E_2, and E_3, whose multiplication on the left of an $r \times c$ matrix M has the same effect as performing the row operations (1), (2), (3) respectively on M. Show that these matrices are invertible.

(3) Show that the number of pivots in any row echelon form of a given matrix M is the same.

(4) Show that reduced row echelon form of a matrix M is uniquely determined.

(5) For what values of c does the system of equations

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \\ c \end{bmatrix}$$

will be consistent? Write the general solution for such a c.

(6) Find the intersection in $\mathbb{R}^3$ of the following planes: $x + 2y + 3z = 1$, $2x + 3y + 5z = 2$, $2x + y + 3z = 2$. Plot this intersection.

(7) Find the reduced row echelon form for the following matrix:

$$\begin{bmatrix} -1 & 1 & -1 & -2 \\ 2 & 0 & 0 & 3 \\ -1 & 0 & -1 & -2 \\ -3 & 0 & 2 & -1 \end{bmatrix}.$$

(8) Find an inverse for the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

using row operations.

(9) Do the polynomials $x^3 + 2x$, $x^2 + x + 1$, $x^3 + 5$, and $x^3 + 3x - 5$ span $\mathbb{P}_3(\mathbb{R})$? (Here we define $\mathbb{P}_3(\mathbb{R})$ to be the vector space of polynomials in the variable x with degree less than or equal to 3.)

(10) Compute the number of pivots in any row echelon form for the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As a result, write the dimensions of its nullspace and range respectively. Can you find any bases for the nullspace and the range?